Infinite Ergodic Theory and Geometry

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Non-Euclidean Geometry

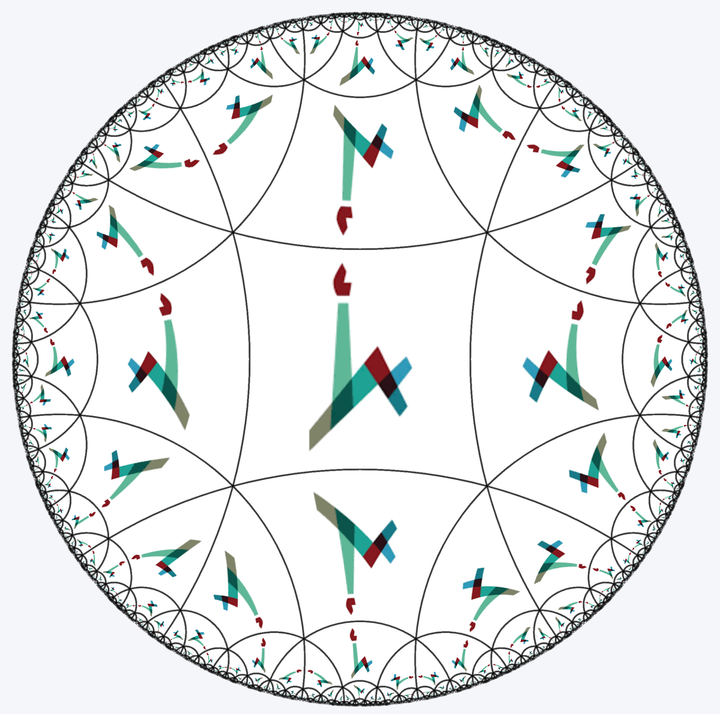
In his classic “the Elements”, Euclid states five postulates describing the foundational assumptions of planar and spatial geometry. Euclid’s fifth axiom, also known as the “parallel” postulate, states the following: **Given a line and a point not on it, exactly one line parallel to the given line can be drawn through the point.**

For more than two millennia mathematicians have believed this fifth postulate is implied by the first four. It was only in the first half of the nineteenth century that new geometric models were discovered, proving the independence of said postulate. These “non-Euclidean” geometries are divided into two fundamental categories:

Elliptic Geometry

Elliptic or spherical, geometry may be modeled on the surface of a sphere in 3D-space. In this model “great circles” play the role of straight lines. Note that in this model, given a line and a point not on it, there are **no lines** parallel to the given line going through the point (parallels are lines which may be continued indefinitely and never meet).

Hyperbolic Geometry



Hyperbolic geometry may be modeled in the interior of a 2D disc. In this model the role of straight lines is played by circular arcs perpendicular to the disc’s boundary. Note that given a line and a point not on it, there exist in this model **infinitely many** lines parallel to the given line and going through the given point.

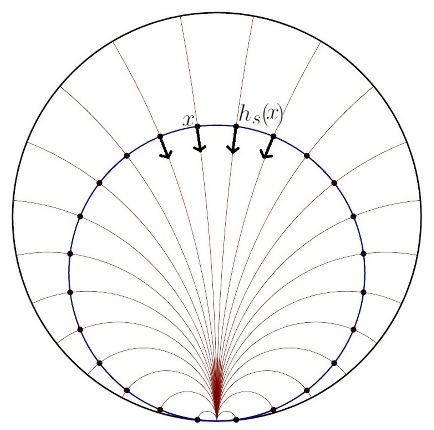
Geodesics and Horocycles

A geodesic between two points on a surface is the path of minimal distance between them. Geodesics are the “straight lines” on curved surfaces. In the hyperbolic plain $\mathbb{H}^2$, given a point p and a direction $\vec{v}$ there is a unique geodesic passing through the point p in the direction $\vec{v}$. One can parameterize this geodesic, along with its tangent vectors, as $a_t(p,\vec{v})$, where $a_0(p,\vec{v}) = (p,\vec{v})$.

The stable horocycle based at $x = (p,\vec{v})$ is the set

$$U_x = \{y \ : \ d(a_t x, a_t y) \to 0, \quad \text{as } t \to \infty\}.$$

Horocycles can be parameterized as well by $h_s(p,\vec{v})$, where $h_0(p,\vec{v}) = (p,\vec{v})$. In the hyperbolic disc model horocycles are circles tangent to the boundary of the disc.



Ergodic Theory

Ergodic theory is the study of long-term statistical behavior of dynamical systems — phase spaces with a notion of time evolution.

A *flow* on a space X is a nice family of functions $\{\varphi_t : X \to X\}$ satisfying

$$\varphi_{t+s}(x) = \varphi_t(\varphi_s(x))$$

for all x in X and real numbers s and t. We may think of φ_t as time evolution maps where $\varphi_t(x)$ is the state of the system at time t after beginning at the initial condition x. An interesting example is the geodesic flow on a hyperbolic surface — X is the phase space of points and directions $(p,\vec{v})$ on some surface $\mathcal{S}$ where $\varphi_t(p,\vec{v})$ denotes the location and direction at time t of a free flying particle with initial conditions $(p,\vec{v})$.

In many cases there exists a notion of a probability measure defined on the space X allowing to draw points at random (e.g. a probability proportional to the volume measure on a surface).

A typical question in ergodic theory is the following:

Given a set E in X, what is the frequency of visits to E of a random trajectory? I.e, given a random x and large T, what is the proportion

$$\frac{1}{T}\left|\{0 \le t \le T \ : \ \varphi_t(x) \text{ is in } E\}\right| ?$$

A probability measure on X is called ***φ-invariant*** if for any nice subset F of X

$$\mathbb{P}[x \text{ is in } F] = \mathbb{P}[x \text{ is in } \varphi_t(F)] \quad \text{for any } t.$$

A probability measure is called ***ergodic*** if for every nice subset F of X the condition that all trajectories beginning in F will forever stay in F implies that $\mathbb{P}[x \text{ is in } F] = 0$ or 1.

We may now state the fundamental theorem of ergodic theory due to Birkhoff, giving an answer to the question above:

The Pointwise Ergodic Theorem (Birkhoff)

Let $\mathbb{P}$ be an invariant and ergodic probability measure on the space X with respect to the flow $\{\varphi_t\}$. For any nice subset E of X

$$\lim_{T\to\infty} \frac{1}{T}\left|\{0 \le t \le T \ : \ \varphi_t(y) \text{ is in } E\}\right| = \mathbb{P}[x \text{ is in } E]$$

for all y of X outside a set of probability 0.

Heuristically, the ergodic theorem states that **time averages and space averages are equal** — sampling a single trajectory over a large period of time is the same as sampling over the whole space.

Infinite Measures and the Ratio Ergodic Theorem

In general, a measurable space may support many different, finite and infinite, invariant and ergodic measures w.r.t a flow $\{\varphi_t\}$. While very useful for finite measures, the pointwise ergodic theorem applied to an infinite φ-invariant and ergodic measure μ, only implies that $\int_0^T \chi_E(\varphi_t(x))dt = o(T)$ for μ-a.e. $x \in X$.

A substitute for Birkhoff’s theorem in the infinite measure context is the following:

The Ratio Ergodic Theorem (Hopf)

Let μ be a conservative, σ-finite, invariant and ergodic measure on the space X with respect to the flow $\{\varphi_t\}$. For any measurable subsets E and F of X

$$\lim_{T\to\infty} \frac{\int_0^T \chi_E(\varphi_t(x))dt}{\int_0^T \chi_F(\varphi_t(x))dt} = \frac{\mu(E)}{\mu(F)}$$

for μ-a.e. x, whenever $\mu(E), \mu(F) < \infty$ and $\mu(F) > 0$.

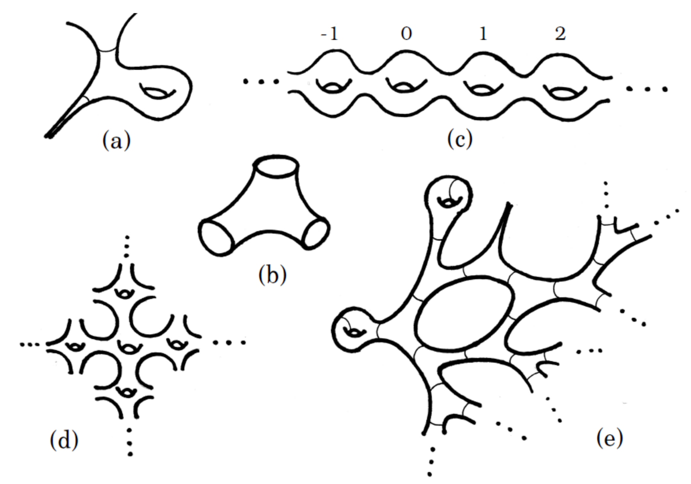
Hence classifying the invariant and ergodic measures (finite or infinite) is closely related to understanding the possible statistical distributions of φ-orbits.

Measure Classification for Horocycle Flows

Classification of horocycle invariant and ergodic locally finite measures on hyperbolic surfaces:

Finite Measures

- **Compact Surface** - Unique ergodicity (Furstenberg)
- **Finite volume** - Unique non-periodic measure (Dani)
- **Infinite volume** - only periodic measures (Ratner)



From Sarig’s Survey paper “Unique Ergodicity for Infinite Measures”

Infinite Measures

- **Geometrically finite Surface** - Unique recurrent measure (Burger, Roblin)
- **Abelian cover of compact surface** - Uncountable family of ergodic recurrent measures (Babillot-Ledrappier)
- **Regular covers of finite volume surface** - All invariant measures are quasi-invariant w.r.t. the geodesic flow (Ledrappier-Sarig)
- **Weakly tame surface** - All invariant measures are quasi-invariant w.r.t. the geodesic flow (Sarig)

Horospherical Flows on Geometrically infinite Manifolds

Denote by $G = SO^+(d,1)$ the group of orientation preserving isometries of hyperbolic d-space. Let $A = \{a_t\}_{t\in\mathbb{R}}$ be the Cartan subgroup of G and let U be the unstable horospherical subgroup with respect to a_1.

Theorem (L - Lindenstrauss)

Let $\Gamma < G$ be any discrete subgroup. Let μ be a U-invariant and ergodic Radon measure on G/Γ. If the set

$$\bigcap_{n=1}^{\infty} \overline{\bigcup_{t>n} a_t g \Gamma g^{-1} a_t}$$

contains a Zariski dense subgroup for μ-a.e. $g\Gamma$, then μ is quasi-invariant w.r.t $N_G(U)$.

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